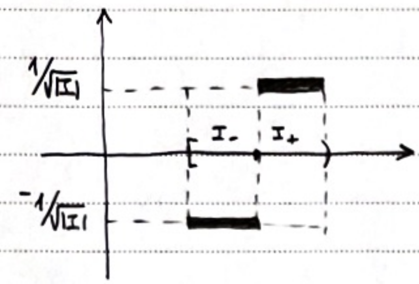
§. Haar Functions ($\mathbb{R}$)

Once we have a dyadic grid $\mathcal{D}$ on $\mathbb{R}$, we associate to every dyadic interval $I \in \mathcal{D}$ the Haar function:

$$\boxed{h_I(x) = \frac{1}{\sqrt{|I|}}\left(\mathbb{1}_{I_+}(x) - \mathbb{1}_{I_-}(x)\right)}$$



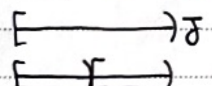
The coefficient $\frac{1}{\sqrt{|I|}}$ is a normalization constant to ensure h_I has L^2-norm 1:

$$\|h_I\|^2_{L^2(\mathbb{R})} = \frac{1}{|I|}\int_{\mathbb{R}} \left(\mathbb{1}_{I_+}(x) - \mathbb{1}_{I_-}(x)\right)^2 dx$$
$$= \frac{1}{|I|}\int_{\mathbb{R}} \left(\mathbb{1}_{I_+}(x) + \mathbb{1}_{I_-}(x)\right) dx$$
$$= \frac{1}{|I|}\left(|I_+| + |I_-|\right) = \underline{\underline{1}}.$$

Remark: Every Haar function has mean 0 ("cancellative").

$$\int h_I(x)\,dx = 0$$

Remark: If $I \subsetneq J$ are dyadic intervals, h_J is constant on I: we denote this value by $h_J(I)$.



$I \subsetneq J \Rightarrow$ either $I \subset J_-$ or $I \subset J_+$

$$I \subset J_- \Rightarrow h_J(x) = -\frac{1}{\sqrt{|J|}},\ \forall x \in I$$
$$I \subset J_+ \Rightarrow h_J(x) = \frac{1}{\sqrt{|J|}},\ \forall x \in I$$

$$h_J(I) = \begin{cases} -\frac{1}{\sqrt{|J|}}, & I \subset J_- \\ \frac{1}{\sqrt{|J|}}, & I \subset J_+ \end{cases}$$

A useful instance where this is used:

$$\langle f \rangle_I = \sum_{J \supsetneq I} (f, h_J)\, h_J(I)$$

Prop.: The Haar functions $\{h_I\}_{I\in\mathcal{D}}$ associated w/ a dyadic grid $\mathcal{D}$ on $\mathbb{R}$ form an orthonormal system in $L^2(\mathbb{R})$: $\|h_I\|_{L^2} = 1,\ \forall I \in \mathcal{D}$ $\quad (h_I, h_J)_{L^2} = \delta_{I,J} = \begin{cases} 0, & \text{if } I \neq J \\ 1, & \text{if } I = J \end{cases}$

Pf.: Verify inner product.

$$(h_I, h_J)_{L^2} = \int_{\mathbb{R}} h_I(x) h_J(x)\,dx$$

- if $I \cap J = \emptyset$: $\boxed{0}$ (disjoint support)
- if $I \cap J \neq \emptyset$:
  - $I = J$: $\|h_I\|^2_{L^2} = \boxed{1}$
  - $I \subsetneq J \Rightarrow h_J$ is constant on I
    $(h_I, h_J) = \int_I h_I(x) h_J(I)\,dx = \boxed{0}$
  - $I \supsetneq J$: same. ■

We will see that this is a complete orthonormal system, i.e. an orthonormal basis for $L^2(\mathbb{R})$.
So we write for $f \in L^2(\mathbb{R})$:

$$f = \sum_{I \in \mathcal{D}} (f, h_I)\, h_I$$

where $(f, h_I) := \int_{\mathbb{R}} f h_I$ is the Haar coefficient of f (wrt I).

$$\langle f \rangle_I = \frac{1}{|I|}\int_I \sum_{J\in\mathcal{D}} (f, h_J) h_J(x)\,dx = \frac{1}{|I|}\left(\sum_{J \subset I} (f, h_J) \underbrace{\int_I h_J}_{0} + \sum_{J \supsetneq I} (f, h_J) \underbrace{\int_I h_J}_{=|I| h_J(I)}\right)$$
$$= \sum_{J \supsetneq I} (f, h_J)\, h_J(I). //$$

9. Completeness of the Haar system in $L^p(\mathbb{R})$:

Theorem: The Haar functions $\{h_I\}_{I\in\mathcal{D}}$ adapted to a dyadic grid $\mathcal{D}$ on $\mathbb{R}$ form an orthonormal basis for $L^2(\mathbb{R})$. More generally, the closure of the linear span of all Haar functions is dense in $L^p(\mathbb{R})$, for all $1<p<\infty$.

Proof.: We need the following important corollary of the Hahn-Banach theorem:

Let $\mathcal{X}$ be a normed vector space and V be a linear subspace of $\mathcal{X}$ (not necessarily closed). Then, for every $x\in\mathcal{X}\setminus\overline{V}$, there is $\varphi\in\mathcal{X}^*$ such that:

$$\varphi(v)=0,\ \forall v\in V \text{ and } \varphi(x)=1, \text{ with } \|\varphi\|_{*}=\frac{1}{\operatorname{dist}(x,V)}$$

So, it suffices to show: $\boxed{\text{Suppose } f\in L^{p'}(\mathbb{R}) \text{ is such that } (f,h_I)=0,\ \forall I\in\mathcal{D}. \text{ Then } f=0.}$ (*)

$\frac{1}{p}+\frac{1}{p'}=1$; $(L^p(\mu))^*=L^{p'}(\mu)$, in the sense that: $\forall$ bounded linear functional $\varphi\in(L^p(\mu))^*$, $\exists! g\in L^{p'}(\mu)$ s.t. $\varphi(f)=(f,g)_{d\mu}=\int fg\,d\mu,\ \forall f\in L^p(\mu)$

This will imply that:

$$\overline{\operatorname{span}\{h_I\}_{I\in\mathcal{D}}}=L^p(\mathbb{R})$$

(what we are trying to prove).

$g\in L^p(\mathbb{R})\setminus\overline{\operatorname{span}\{h_I\}_{I\in\mathcal{D}}} \Rightarrow \exists f\in L^{p'}(\mathbb{R})$ s.t. $(f,h_I)=0,\ \forall I\in\mathcal{D}$; $(f,g)=1$.

But by (*), $f=0 \Rightarrow$ cannot have $(f,g)=1 \Rightarrow L^p(\mathbb{R})\setminus\overline{\operatorname{span}\{h_I\}_{I\in\mathcal{D}}}=\emptyset$.

To prove (*), we show that:

$$\boxed{f\in L^{p'}(\mathbb{R});\ (f,h_I)=0,\ \forall I\in\mathcal{D}} \Rightarrow \boxed{\langle f\rangle_I=0,\ \forall I\in\mathcal{D}}\ (**)$$

$$\Downarrow$$

$$\boxed{\langle f\rangle_U=0,\ \forall\, U=(a,b) \text{ open interval in } \mathbb{R}}$$

Because: Every $U=(a,b)$ is the disjoint union of the maximal dyadic intervals contained in U.

$$\Downarrow \text{ Lebesgue Differentiation Theorem}$$

$$\boxed{f=0 \text{ a.e.}}$$

$\forall f$ locally integrable on $\mathbb{R}^n$:

$$\lim_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy=f(x) \text{ a.e. } x\in\mathbb{R}^n$$

(will see this soon, w/ the maximal function)

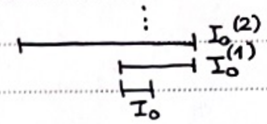
$\mathcal{D}_U:=\{\text{maximal } I\in\mathcal{D} \text{ s.t. } I\subset U\}$

- "maximal" means: $\nexists J\in\mathcal{D}$ s.t. $I\subsetneq J$ & $J\subset U$
- obvious: $\bigcup_{I\in\mathcal{D}_U} I\subset U$.
- converse: let $x\in U\subset\mathbb{R} \Rightarrow \exists I\in\mathcal{D}$ s.t. $x\in I\subset U$ $\Rightarrow \exists \tilde{I}\in\mathcal{D}_U$ (maximal) s.t. $I\subset\tilde{I}$
- DISJOINTNESS: Suppose $I,J\in\mathcal{D}_U$ are maximal dyadic intervals in U. If $I\cap J\neq\emptyset$, then either
  - $I\subsetneq J \Rightarrow$ contradicts maximality of I in $\mathcal{D}_U$
  - $I\supsetneq J \Rightarrow$ contradicts maximality of J in $\mathcal{D}_U$
  - $I=J$ only possibility

<u>Proof of (**)</u> : $f \in L^{p'}(\mathbb{R}), (f, h_I)=0, \forall I \in \mathcal{D} \Rightarrow \langle f \rangle_I = 0, \forall I \in \mathcal{D}$.

Let $f \in L^{p'}(\mathbb{R})$ be such that $(f, h_I)=0, \forall I \in \mathcal{D}$, and <u>fix</u> $I_0 \in \mathcal{D}$. <u>Show</u> : $\boxed{\langle f \rangle_{I_0} = 0}$

For each $k>0$ let $I_0^{(k)}$ be the k^{th} dyadic parent of I_0.



$$(f, h_{I_0^{(k)}})=0 \Rightarrow \int_{(I_0^{(k)})_-} f\,dx = \int_{(I_0^{(k)})_+} f\,dx$$

$$= \int_{I_0^{(k-1)}} f\,dx$$

(one of $(I_0^{(k)})_-, (I_0^{(k)})_+$ <u>is</u> $I_0^{(k-1)}$).

$$\Rightarrow \langle f \rangle_{I_0^{(k)}} = \frac{1}{|I_0^{(k)}|} \int_{I_0^{(k)}} f\,dx = \frac{1}{|I_0^{(k)}|} \cdot 2 \int_{I_0^{(k-1)}} f\,dx = \langle f \rangle_{I_0^{(k-1)}}$$

$$\Rightarrow \boxed{\langle f \rangle_{I_0^{(k)}} = \langle f \rangle_{I_0^{(k-1)}}, \ \forall k \geq 1} \text{ i.e. } \langle f \rangle_{I_0} = \langle f \rangle_{I_0^{(1)}} = \langle f \rangle_{I_0^{(2)}} = \dots$$

Hölder inequality: $$\left| \int_{I_0^{(k)}} f\,dx \right| \leq \int_{\mathbb{R}} |f| \cdot \mathbb{1}_{I_0^{(k)}}\,dx \leq \left(\int_{\mathbb{R}} |f|^{p'} dx \right)^{1/p'} \left(\int \mathbb{1}^p_{I_0^{(k)}}\,dx \right)^{1/p}$$

$$= \|f\|_{L^{p'}(\mathbb{R})} \, |I_0^{(k)}|^{1/p}$$

$$\Rightarrow |\langle f \rangle_{I_0}| = |\langle f \rangle_{I_0^{(k)}}| = \frac{1}{|I_0^{(k)}|} \left| \int_{I_0^{(k)}} f\,dx \right|$$

$$\leq \frac{1}{|I_0^{(k)}|} \|f\|_{L^{p'}(\mathbb{R})} |I_0^{(k)}|^{1/p} = \frac{\|f\|_{L^{p'}(\mathbb{R})}}{|I_0^{(k)}|^{1/p'}} \xrightarrow[k \to \infty]{} 0$$

$$\Rightarrow \langle f \rangle_{I_0} = 0 \ //$$

■

Remarks :

$\boxed{1.}$ **The statement is not true for $L^1(\mathbb{R})$:**

The map Λ that takes the mean of a function $\Lambda f := \int_{\mathbb{R}} f \, dx$ is a continuous linear functional on $L^1(\mathbb{R})$. This map vanishes on all Haar functions: $\Lambda h_I = 0, \ \forall I \in \mathcal{D}$, but Λ is clearly not the zero map! ($\exists$ functions in $L^1(\mathbb{R})$ w/ non-zero mean)

$\boxed{2.}$ **The statement is also false for $L^\infty(\mathbb{R})$:**

The Haar functions are compactly supported, and the compactly supported functions are not dense in $L^\infty(\mathbb{R})$.

$\boxed{3.}$ **The set of Haar functions $\{h_I\}_{I\in\mathcal{D}}$ is also a minimal set of functions such that the closure of their linear span is dense in L^p:** if we remove one Haar function h_{I_0} from this set, then $h_{I_0} \in L^{p'}(\mathbb{R})$ is a non-zero linear functional that vanishes on all other Haar functions, which therefore no longer span a dense subspace. In this sense, $\{h_I\}_{I\in\mathcal{D}}$ is a basis for $L^p(\mathbb{R})$.

→ Let $\mathcal{H}$ denote the linear span of $\{h_I\}_{I\in\mathcal{D}}$. We have that $\overline{\mathcal{H}} = L^p(\mathbb{R})$, $1<p<\infty$.
Let $f \in L^p(\mathbb{R})$ and suppose $f_n \in \mathcal{H}$ converge to f in L^p as $n \to \infty$. For each n, write

$$f_n := \sum_{I\in\mathcal{D}} a_I^{(n)} h_I, \text{ where } a_I^{(n)} \neq 0 \text{ only for finitely many } I.$$

Then

$$\lim_{n\to\infty} a_I^{(n)} = \lim_{n\to\infty} \Big(\sum_{J\in\mathcal{D}} a_J^{(n)} h_J, h_I \Big) = (f, h_I)$$

=> we would like to write $\boxed{f = \sum_{I\in\mathcal{D}} (f, h_I) h_I}$

→ in L^2, this sum is easily interpreted and well-behaved, using Hilbert space techniques;

→ in L^p, convergence of this infinite sum is not so clear (for example, it may depend on the order of summation.)

Address this soon: Haar multipliers & Martingale Transform

Remark: Mentioned earlier: $\langle f\rangle_I = \sum_{J \supsetneq I} (f, h_J) h_J(I)$

Obtained how? $$f = \sum_{J\in\mathcal{D}} (f, h_J) h_J$$

$$\int_I f = \int_I \sum_{J\in\mathcal{D}} (f,h_J) h_J$$

$$\langle f\rangle_I = \sum_{J\in\mathcal{D}} (f,h_J) \langle h_J\rangle_I \quad \to \begin{cases} 0 & \text{if } J\subseteq I \\ h_J(I) & \text{if } J\supsetneq I\end{cases}$$

$$\Rightarrow \boxed{\langle f\rangle_I = \sum_{J\supsetneq I} (f,h_J) h_J(I)}$$

Another useful formula: $$\boxed{\frac{\mathbb{1}_I}{|I|} = \sum_{J\supsetneq I} h_J(I) h_J}$$

$$\frac{\mathbb{1}_I}{|I|} = \sum_J \frac{1}{|I|} \underbrace{(\mathbb{1}_I, h_J)}_{\substack{0 \text{ if } J\subseteq I \\ h_J(I)|I| \text{ if } J\supsetneq I}} h_J = \sum_{J\supsetneq I} h_J(I) h_J$$

So what about this argument: (staying in L^2 for now)

$$f = \sum_I (f,h_I) h_I$$

$$\Rightarrow \int_{\mathbb{R}} f = \int_{\mathbb{R}} \sum_I (f,h_I) h_I$$

$$= \sum_I \underbrace{\int_{\mathbb{R}} (f,h_I) h_I}_{0} = 0 \Rightarrow \text{all } L^2 \text{ functions have mean } 0\,?!$$

Several issues here: 1. interchanging Σ and $\int$ $\to$ only OK over compact sets assuming uniform convergence

2. integrating over $\mathbb{R}$ $\to$ $\Lambda: f \mapsto \int_{\mathbb{R}} f$ is ~~not~~ a bounded linear functional on L^2!

Why are the computations above OK? Because everything is over a compact set!

More on this next...

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